### Section 4.5: Row and column spaces

New vocabulary:

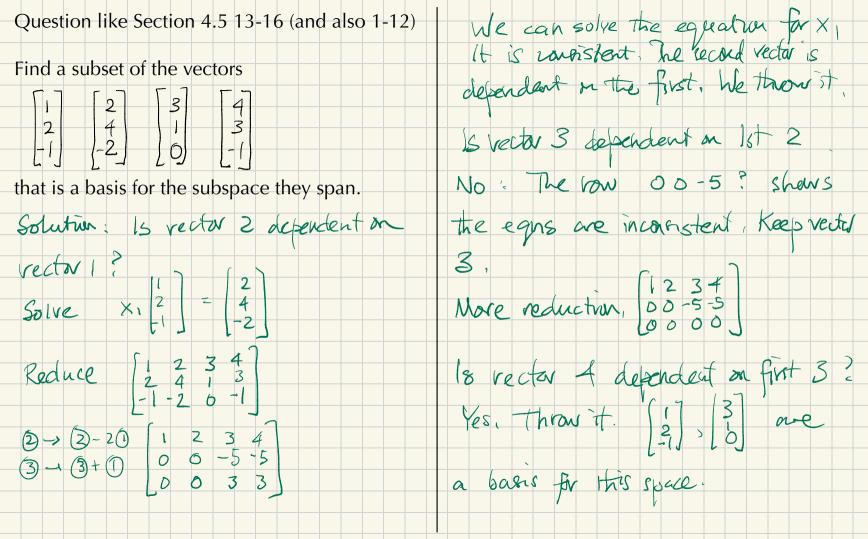
- row space, column space of a matrix
- row rank, column rank, rank
- pivot column
- null space

We learn:

- an algorithm to find a basis for the column space
- an algorithm to find a basis for the row space
- how to find a subset of a spanning set that is a basis
- how to extend an independent set to a basis

Question like Section 4.5 13-16 (and also 1-12)

Find a subset of the vectors	(as the 3rd rectar dependent on the
$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 1 & 2 & 6 & 1 \end{bmatrix}$	first two: les - throw it
$\begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} -2 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} -1 \end{bmatrix}$ that is a basis for the subspace they span.	No - keept.t,
Overview of the algorithm.	We end with a set of vectors that
Take the first vector (Fit is non-zero,	is independent with the some span
keep it.	al the original rectars.
Is the second rectar dependent on the	
rectors we have so far (i.e. a scalar	
multiple of these first).	
(F dependent, throw "it out (F not, keep it	



## Definition: the column space of a matrix A is is the span of the columns of the matrix

Example: the column space of

3 -2 0 -1

is the space spanned by the vectors 

Definition: a pivot column of a matrix A is a column of A for which the echelon form of A has a leading entry (or pivot). 1 2 3 Example: The pivot columns of 2 4

Theorem.

The pivot columns of a matrix A form a basis for the column space of A. ER

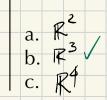
## Done

Example: The vectors  $\begin{bmatrix} 1\\2\\-1\end{bmatrix}$ are a basis for the column space of A.

## **Questions:**

1. What is the dimension of the column space of the matrix A in the example? The Gais has 2 elements. a. 2 🗸

2. Of what space is the column space of A a subspace?



b. 3 c. 4

4

3

-1

-2

- 1

0

# e-class Warm-ub!!!

What is the dimension of the column space of the following matrix?

b.

e. 4

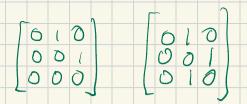
When did we learn the meaning of the word basis for a vector space V?

a. This week

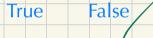
Last week Ь

Definition. The row space of a matrix A is the Theorem 2. The row space of A is not span of the rows of A. changed by elementary row operations. Hence 2 4 4 3 A has the same row space as its echelon form. 2 Example. The row space of -2 0 -1 c(12]+d[0-2] is the subspace of  $R^4$  spanned by These two sels of rectars are the same 1234 [2413] and -1-20-1 because (D -2] = [3 4] - 3 [1 2] In the look the row space is spanned by  $(1234) = \frac{2}{3}$ ,  $(2413) = \frac{4}{3}$  and  $(-1-\overline{20})$ Look at the row space of1(It's quicker to write down!)3 = (c - 3d) [1, 2] + d (34)Similarly we can write a [12] + 6[34] It is the set of all vectors as a linear combin of [12], [0-2] because 9 [1 2] + 6 [3 4] where a, b e R [3,4] = [0,-2] + 3[12]Put it in eche on - Ann 3-30-30 12 a[12] + b[34] = a[12] + b[0, -2] + 3[1, 2])This has now space all rectors

Question: true or false in general?:



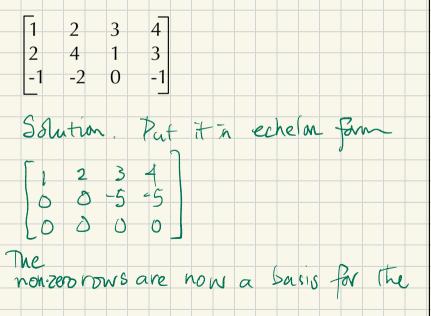
'Let A be a matrix. The column space of A is not changed by elementary row operations. Hence A has the same column space as its echelon form.'



Theorem 2. The row space of A is not changed by elementary row operations. Hence A has the same row space as its echelon form.

Question like Section 4.5, 1-12

Find a basis for the row space of the matrix



tow space. They span (row space is unchanged) They are independent: if a (1234) + b (00-5-5) = (0003) (a, 2a, 3a-5b, 4a-5b) ~ (0000) a = 0, 3a - 5b = 0, b = 0. This now space how dimension 2 as a subspace of Rª

Theorem 2 extra: The non-zero rows of the echelon form of A form a basis for the row space of A

Definition. The row rank of a matrix A is the dimension of its row space.

The column rank of a matrix A is the dimension of its column space.

Theorem. For any matrix A the row rank and column rank are equal.

Example. For the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ -1 & -2 & 0 & -1 \end{bmatrix}$ 

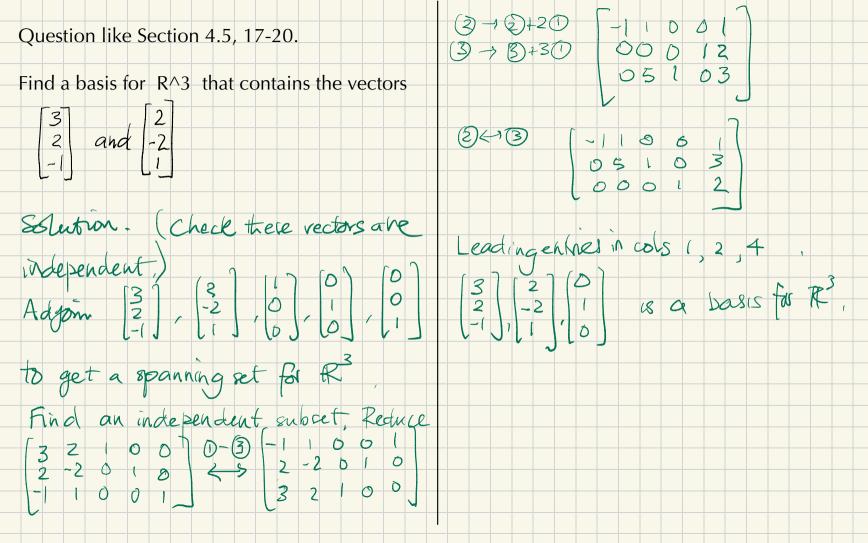
Definition. The common value of the row rank and column rank of a matrix is called the rank of the matrix.

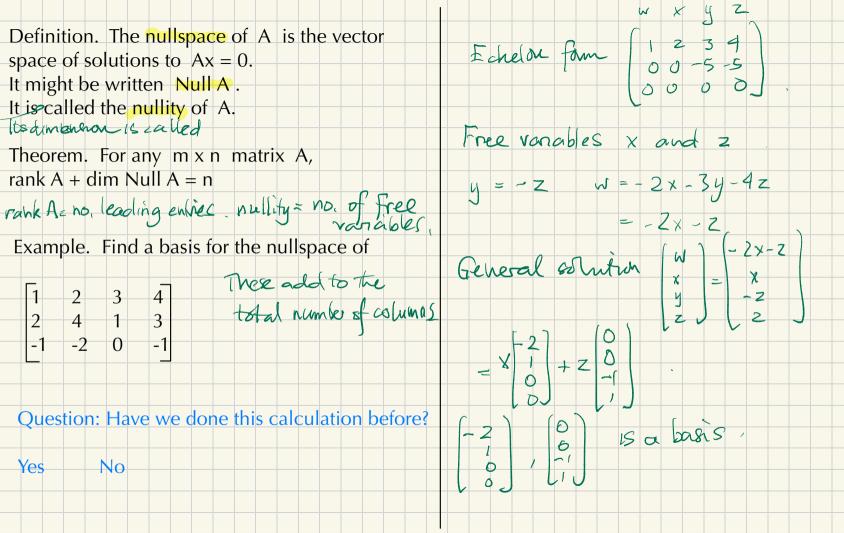
# Proof of theorem

rows in the echelor pm.

Lolumn rank = number of leading entree in the echelon Form.

These are the same





Definition. The nullspace of A is the vector space of solutions to Ax = 0. It might be written Null A. It is called the nullity of A.

Theorem. For any  $m \ge n$  matrix A, rank A + dim Null A = n

Example. Find a basis for the nullspace of

 $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ -1 & -2 & 0 & -1 \end{bmatrix}$ 

#### **Ouestions:**

Are the following true or false in general for an n x n matrix A?

a. If A has rank n then Ax = b has a solution for every vector b in  $R^n$ .

Truě False Not enough information to say.

b. If Ax = 0 has a unique solution then Ax =b always has a solution, for every b in  $R^n$ .

Not enough information to False True say.

unique solution => Null A = {0] so rank A = n, = dimension of

col. space. Every & lies in the column. Ax=5 has a solution.

Question:

Suppose that A is an m x n matrix (m rows, n columns) and that Ax = 0 has a unique solution. Which of the following statements is sometimes false?

a. The columns of A are linearly independent.

b. The columns of A span a space of dimension n.

c. The columns of A are a basis for the space they span.

d. m≤n